C. Streeter-Phelps Equation

The Streeter-Phelps equation is the solution to the differential equation

$$\frac{dD}{dt} = k_d L_0 \exp(-k_d t) - K_r D, \tag{C.1}$$

derived in Section 5.2.2. The oxygen deficit is $D = [O_2]_{sat} - [O_2]$, k_d is the degradation rate of organic matter, L_0 is the total carbonaceous oxygen demand, and K_r is the river oxygen aeration coefficient. The solution is subject to the initial condition $D(t = 0) = D_0$.

Since this is an inhomogeneous equation, we first find the complimentary solution, which is the solution to the homogeneous equation

$$\frac{dD}{dt} = -K_r D, \tag{C.2}$$

which has the solution

$$D_c(t) = C_1 \exp(-K_r t), \tag{C.3}$$

where C_1 is a constant that must satisfy the initial condition in the final solution.

To find a particular solution, we assume the solution has the same form as the forcing function ($k_d L_0 \exp(-k_d t)$). Thus, we assume the solution

$$D_p(t) = A \exp(-k_d t). \tag{C.4}$$

Substituting into (C.1) and solving for A, we obtain

$$A = \frac{k_d L_0}{K_r - k_d}. \tag{C.5}$$

The general solution is the sum of the complimentary and particular solutions:

$$D(t) = \frac{k_d L_0}{K_r - k_d} \exp(-k_d t) + C_1 \exp(-K_r t). \tag{C.6}$$

Setting $t = 0$ and equating with the initial condition, leads to

$$C_1 = D_0 - \frac{k_d L_0}{K_r - k_d}. \tag{C.7}$$

Substituting this result into the general solution yields the classic Streeter-Phelps equation:

$$D(t) = \frac{k_d L_0}{K_r - k_d} (\exp(-k_d t) - \exp(-K_r t)) + D_0 \exp(-K_r t). \tag{C.8}$$