MASS TRANSPORT PROCESSES

When a tracer cloud is introduced into a fluid flow (e.g. river) two processes are visible:

- The cloud is carried away from the point of discharge by the mean current.
- The cloud spreads in all directions due to irregular motions and grows in size.

**Advection** - mass transport by mean velocity field

**Diffusion** - mass transport by irregular fluctuations ("random movements")

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PASSIVE MIXING PROCESSES - advection + diffusion as they exist in environment (passive source)

ACTIVE MIXING PROCESSES - generation of mean and random velocity field by source - momentum or buoyancy
MOLECULAR DIFFUSION

- molecular diffusion of dissolved or suspended matter is caused by random movement through molecular motions of individual molecules

\[ \frac{\partial c}{\partial x} < 0 \]

\[ J_x = \text{diffusive mass flux} \cdot \frac{\text{mass}}{\text{area, time}} = \frac{M}{L^2, T} \]

**Fick’s Law of Diffusion:**

- rate of transport is proportional to the spatial concentration gradient \( \frac{\partial c}{\partial x} \)

\[ J_x = -D_{AB} \frac{\partial c}{\partial x} \quad \text{Fick’s law (1-D)} \]

\( D_{AB} = \text{coefficient of molecular diffusivity} = \frac{L^2}{T} = f \text{ (solvent, solute, temperature...)} \)

Typical values for molecular diffusion:

**Dissolved matter in water:** \( D \sim 2 \times 10^{-5} \text{ cm}^2/\text{s} \)
**Gases in air:** \( D \sim 2 \times 10^{-1} \text{ cm}^2/\text{s} \)

**Generalization:**

\[ \vec{J} = (J_x, J_y, J_z) \]

\( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \) divergence vector

\[ \vec{J} = -D \nabla c \quad \text{Fick’s law (3-D)} \]

**Example:** Salt diffusion in a vessel

- initial sharp interface
- two hours later diffuse interface
- \( \sim 1.0 \text{ cm} \)
ADVECTIVE MASS FLUX

- transport of matter due to mean motion (current)

\[ \bar{q} = (u, v, w) = \text{hydrodynamic velocity} \]

\[ c\bar{q} = \begin{bmatrix} M \\ L^3 \\ T \end{bmatrix} = \begin{bmatrix} M \\ L^2 \\ T \end{bmatrix} = \text{advective mass flux} \]

Example:

Pure advection in x-direction (no diffusion no reaction)

TOTAL MASS FLUX:  Advection + diffusion

\[ \bar{N} = (N_x, N_y, N_z) = c\bar{q} + \bar{J} = c\bar{q} - D\nabla C \]
MASS CONSERVATION PRINCIPLE

R = rate of production or decay per unit volume = \[ \frac{M}{L^3, T} \]

\[
\begin{align*}
\text{net flux of mass} & \quad + \quad \text{Rate of production/decay of mass} \\
\text{(In – Out)} & \quad + \quad \text{(chemical, physical, biological...)} \\
& \quad = \quad \text{Rate of mass accumulation within element}
\end{align*}
\]

\[- \frac{\partial N_x}{\partial x} \Delta x \Delta y \Delta z - \frac{\partial N_y}{\partial y} \Delta x \Delta y \Delta z - \frac{\partial N_z}{\partial z} \Delta x \Delta y \Delta z + R \Delta x \Delta y \Delta z = \frac{\partial c}{\partial t} \Delta x \Delta y \Delta z \]

\[
\frac{\partial c}{\partial t} + \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} = \frac{\partial c}{\partial t} + \nabla \cdot \vec{N} = \frac{\partial c}{\partial t} + \nabla \cdot \vec{q} - \nabla \cdot (D \nabla c) = R
\]

\[
\frac{\partial c}{\partial t} + \vec{q} \cdot \nabla c = D \nabla^2 c + R
\]

Convective Diffusion Equation

\[
\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} = D \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) + R
\]

Temporal change

Adective transport

Diffusive transport

Reaction
Solution of the convective diffusion equation:

1. Basic solution: Instantaneous point source (IPtS) in uniform current and infinite space

\[ \frac{\partial c}{\partial t} + U \frac{\partial c}{\partial x} = D \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) - kc \]

Initial Conditions (I.C.) \[ t < 0, \ c = 0 \quad t = 0 \quad \text{Sudden mass input} \ M \text{ on } (x_1, y_1, z_1) \]

Boundary Conditions (B.C.) \[ c \to 0 \text{ as } (x, y, z) \to \infty \]

Solution:

\[ c = \frac{M}{(4\pi Dt)^{3/2}} \cdot e^{-\frac{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}{4Dt}} \cdot e^{-kt} \]

1. Position of cloud center:
   \[ (x_1 + Ut, y_1, z_1) \]

2. Cloud size:
   \[ 4Dt = 2\sigma^2, \quad \sigma = \text{variance} \]
   \[ \sigma = \sqrt{2Dt} = \text{standard deviation} \sim t^{0.5} \]

3. Maximum concentration
   \[ c_{\text{max}} \sim t^{-3/2} \]
2. Other solutions

Boundary conditions (finite space):

on $\Gamma$: $c = \text{specified}$

Ex: $c = 0$ Absorption condition (Dirichlet B.C.)

$$\frac{\partial c}{\partial n} = \text{specified}$$

Ex: $\frac{\partial c}{\partial n} = 0$ Diffusion barrier (Reflection condition) (Neumann B.C.)

Multiple sources $\rightarrow$ superposition, due to linearity of governing equation

$\rightarrow$ image sources

Time dependence

Initial condition:

- Instantaneous sources
- Continuous sources
TURBULENCE - an instability of a base flow (shear flow)

Possibility 1:
- damping of disturbance (viscosity)
- stable flow
LAMINAR FLOW

Possibility 2:
- insufficient damping
- large eddies
- secondary eddies
TURBULENT FLOW

SPECTRUM OF EDDIES

Large eddies: Integral length scale \( \ell_1 \) => Integral time scale \( t_i = \frac{\ell_1}{u_1} \)

Extraction of energy from mean flow: \( \varepsilon \sim \frac{K.E.}{time} \) \( \approx \frac{u_i^2}{\ell_1/u_1} \)

Energy cascade (transfer of energy from larger to smaller scales)

Smallest eddies: at some scale (Kolmogorov length scale) the eddy size will be small enough so that viscous damping will prevent further breakdown

Dimensionality: viscosity \( \nu = \left[ \frac{L^2}{T} \right] \), energy dissipation rate \( \varepsilon = \left[ \frac{L^2}{T^3} \right] \)

Kolmogorov length scale \( \eta = \frac{L^{3/4}}{\varepsilon^{1/4}} \)

Turbulence spectrum: \( \ell_1 \) "equilibrium subrange"

Overall criterion: Reynolds number \( Re = \frac{UL}{\nu} > 100 \) to 1000
U ~ base velocity, L ~ base flow length
TURBULENT DIFFUSION

Random motion due to turbulent velocity fluctuations cause additional transport ("small scale advection").

\[ u' (t) = \text{fluctuating velocity} \]

\[ \bar{u}(t) = \frac{1}{T} \int_{t}^{t-T} u(t') \, dt' = \text{mean velocity} \]

\[ \bar{u}' = \frac{1}{T} \int_{t}^{t-T} u'(t') \, dt' = 0 \]

\[ u = \bar{u} + u' \quad \text{w} = \bar{w} + w' \]

Averaging process:

\[ v = \bar{v} + v' \quad c = \bar{c} + c' \]

Reynolds decomposition

\[ \bar{uc} = \bar{u}\bar{c} + u'\bar{c} + \bar{u}'c' \]

\[ 1 \int_{t}^{t+T} \left[ \frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} + \frac{\partial vc}{\partial y} + \frac{\partial wc}{\partial z} = D \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) + R \right] \, dt \]

\[ \frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} + \bar{w} \frac{\partial \bar{c}}{\partial z} = - \frac{\partial}{\partial x} \bar{u}c' - \frac{\partial}{\partial y} \bar{v}c' - \frac{\partial}{\partial z} \bar{w}c' + D \left( \frac{\partial^2 \bar{c}}{\partial x^2} + \frac{\partial^2 \bar{c}}{\partial y^2} + \frac{\partial^2 \bar{c}}{\partial z^2} \right) + R \]

\[ \bar{J}_l = (\bar{u}c', \bar{v}c', \bar{w}c') = \text{turbulent diffusive mass flux} = \left[ \frac{M}{L^2 T} \right] \]

\[ \bar{J}_l \text{ needs parameterization : analogy to molecular diffusion (Fick's law)} \]
Example: Tank with Grid Stirring

\[ \text{net transport } \overline{w'c'} > 0 \]

**DIFFUSION IN TURBULENT FLOW**

→ fluctuating velocity field ⇒ multiple scale eddies (spectrum)!

Ex: Release from POINT SOURCE

\[ \sigma_x \sim \sqrt{u'^2} t \sim t^{1/2} \]

Rapid growth (Taylor theorem, 1921)

\[ \frac{\sigma}{u'^2} \approx \text{rms velocity} - u_i \]

a) Short time after release \( t < t_i, \sigma < \ell_1 \)
First small eddies, then eddies of increasing size will cause spreading

b) Long time after release \( t > t_i \)
Eddies are independent of each other

\[ \sigma_x = \sqrt{2(u_i^2 t_i)} \sim t^{1/2} \]

"Fickian" behavior: corresponds to spreading with gradient-type diffusion and a constant coefficient

\[ J_{lx} = u'c' = -E_x \frac{\partial \overline{c}}{\partial x}, \quad J_{ly} = v'c' = -E_y \frac{\partial \overline{c}}{\partial y}, \quad J_{lz} = w'c' = -E_z \frac{\partial \overline{c}}{\partial z} \]

\( u_i \) = turbulent diffusivity = \( \frac{L^2}{T} \)
PRACTICAL TURBULENT DIFFUSION EQ.: \[ \bar{c} \rightarrow c \]

\[
\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} = \frac{\partial}{\partial x} \left( E_x \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( E_y \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( E_z \frac{\partial c}{\partial z} \right) + R + D \nabla^2 c
\]

valid for \( t > t_1 \), \( \sigma > \ell_1 \)

Diffusivities can have dependence on spatial position and direction:

- \( E_x, E_y, E_z = f(x, y, z) \) non-homogeneous

if \( E_x, E_y, E_z = \text{const.} \) homogeneous \( \rightarrow E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} + E_z \frac{\partial^2 c}{\partial z^2} \)

- \( E_x \neq E_y \neq E_z \) anisotropic

if \( E_x = E_y = E_z = E \) isotropic \( \rightarrow E \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) \)

Estimation of turbulent diffusivities:

\[ E = f \] (base flow characteristics)

Order of magnitude:

\[ E \sim U_{t1} \ell_{t1} \text{ single scale only!} \]

TURBULENT CHANNEL FLOW

\[ \text{Re} = \frac{Uh}{\nu} > 500 \text{ for turbulence} \]

\[ \tau_0 = \rho ghS = \text{shear stress} \]

\[ \sqrt{\frac{\tau_0}{\rho}} = u_* = \text{friction (shear) velocity} \]

\[ u_* = \sqrt{ghS} \]

Dominating scales:

\[ u_1 \sim u_* \ell_1 \sim h \]

Also:

\[ \tau_0 = \rho U^2 \]

\[ f = \text{Darcy-Weisbach coefficient, } 0.02 \ldots 0.1 \]

\[ u_* = \sqrt{\frac{f}{8} U^2} \]

Or:

\[ U = \frac{1}{n} h^{2/3} S^{1/2} \]

\[ n = \text{Mannings's } n, \text{ } 0.02 \ldots 0.05 \]

\[ u_* = \sqrt{\frac{gn^2}{h^{1/3} U^2}} \]

Rule of thumb: \( u_* \approx (0.05 \text{ to } 0.1) U \)
**MIXING IN RIVERS**

**Eddy diffusivity \( E_z \)**

We expect: \( E_z \sim u_*, h \)

Velocity profile

\[
\frac{du}{dz} = f(u_*, z)
\]

Dimensional analysis:

\[
\frac{du}{dz} \sim \frac{u_*}{z} \quad \text{or} \quad \frac{du}{dz} = \frac{u_*}{\kappa z}
\]

Upon integration:

\[
\frac{u}{u_*} = \frac{1}{\kappa} \ln z + C_1 \quad \text{log-law}
\]

\[\kappa = \text{von Karman constant} = 0.4\]

\[C_1 = f(\text{Re}, \text{roughness})\]

\[
\tau = \tau_0 \left( 1 - \frac{z}{h} \right) = \rho \varepsilon_z \frac{du}{dz}
\]

\[\varepsilon_z = \text{eddy viscosity} = \left[ \frac{L^2}{T} \right]\]

Model for momentum exchange

(Boussinesq)

\[
\frac{\varepsilon_z}{u_*, h} = \kappa \frac{z}{h} \left( 1 - \frac{z}{h} \right) \equiv \frac{E_z}{u_*, h}
\]

Reynolds analogy for turbulent flows

Mass exchange \( \approx \) momentum exchange

Depth averaged:

\[
\overline{E_z} = \frac{\kappa}{6} = 0.067 \approx 0.1
\]

**Rules of thumb:**

River \( U, h \quad u_* \approx 0.1U \)

\( E_z \approx 0.1u_*, h \approx 0.01Uh \)

Ex:

\( U = 2 \text{ m/s}, \ h = 2\text{m} \)

\( E_z = 0.04 \text{ m}^2/\text{s} = 400 \text{ cm}^2/\text{s} \gg D = 2 \times 10^{-5} \text{ cm/s} \) !!
**VERTICAL MIXING**

e.g. source at surface

\[ \sigma_z = \sqrt{2E_z t} = \sqrt{2E_z \frac{x}{U}} \]

Complete vertical mixing: \( 2.15 \sigma_z = h \) 10% criterion with Gaussian profile

\[
\frac{h}{2.15} = \sqrt{2E_z \frac{x_m}{U}} \quad \Rightarrow \quad x_m = \left( \frac{h}{2.15} \right)^2 \frac{U}{2E_z}
\]

Ex: \( x_m = \left( \frac{2}{2.15} \right)^2 \frac{2}{2 \times 0.04} = 25 \text{ m} \) Very fast

Typically: Distance to complete vertical mixing 10 to 20 water depth!

**LATERAL MIXING**

Wide Channel, Depth \( h \)

Assume isotropy: \( E_y \approx \overline{E_z} = 0.067 u, h \)

Data: \( \frac{E_y}{u, h} \approx 0.15 \) to 0.20 straight, uniform channels

Evaluation of field / laboratory tests:

\[
\frac{1}{2} \frac{d}{dt} \sigma_y^2 = E_y \approx \frac{1}{2} \frac{\sigma_y^2(t_2) - \sigma_y^2(t_1)}{t_2 - t_1}
\]

or

\[
\frac{1}{2} \frac{\sigma_y^2(x_2) - \sigma_y^2(x_1)}{(x_2 - x_1)/U}
\]

Outlines of tracer cloud

Data from Mississippi River study on longitudinal mixing (after McQuivey and Keefer 1976b)
NATURAL CHANNELS; IRREGULARITIES

Data: \( \frac{E_y}{u_h h} = 0.5 \text{ to } 1.0 = \alpha \quad \alpha = 0.6 \pm 50\% \quad (\text{Fischer (1972)}) \)

\[ \alpha = 0.4 \left( \frac{U B}{u_h R_c} \right)^2 \quad \text{Yotsukura and Sayre (1976)} \]

Cumulative Discharge Method \( \text{Yotsukura and Cobb (1976)} \)

\[ dq = h \ddot{u} dy = \text{local discharge} \rightarrow \text{from measurement or numerical model} \]

\[ q(y) = \int_0^y h \ddot{u} dy = \text{cumulative discharge} \]

\[ Q = \int_0^B h \ddot{u} dy = \text{total discharge} \]

Approximate local balance:

\[ \ddot{u} (y) h(y) \frac{\partial c}{\partial x} = \frac{\partial}{\partial y} \left[ E_y (y) h(y) \frac{\partial c}{\partial y} \right] \]

\[ \text{depth integrated advection} \quad \text{depth integrated lateral diffusion} \]

\[ \ddot{u} h \frac{\partial c}{\partial x} = \ddot{u} h \frac{\partial}{\partial q} \left( h^2 E_y \ddot{u} \frac{\partial c}{\partial q} \right) \quad \Rightarrow \quad \frac{\partial c}{\partial x} = \frac{\partial}{\partial q} \left( h^2 E_y \ddot{u} \frac{\partial c}{\partial q} \right) \]

"Diffusivity" = \[ \frac{L^5}{T^2} \approx \text{constant} \]

Understanding of advective velocity field is key to many environmental diffusion problems!
Transverse distributions of dye observed in the Missouri River near Blair, Nebraska, by Yotsukura et al. (1970), plotted (a) versus actual distance across the stream and (b) versus relative cumulative discharge [Yotsukura and Sayre (1976)].
LONGITUDINAL DISPERSION (SHEAR FLOW DISPERSION)

The stretching of a tracer cloud due to vertical and horizontal velocity shear is called longitudinal dispersion. In this case, the velocity differences interact with the transverse diffusion effect and produce an additional transport mechanism.

Observations: $t > t_{mixing}$ or $\sigma_z > h$

For long time:
1. Gaussian distribution
2. $\sigma_L = \sqrt{t}$

Therefore: behavior is analogous to Fickian diffusion.

Analysis:

$\frac{\partial c}{\partial t} + u(z,t) \frac{\partial c}{\partial x} = \frac{\partial}{\partial x} \left( E_x \frac{\partial c}{\partial z} \right) + \frac{\partial}{\partial z} \left( E_z \frac{\partial c}{\partial z} \right)$

Spatial decomposition: $u = U + u''$, $c = C + c''$, $u''$, $c'' = $ spatial deviations

Substitute and average $\frac{1}{h} \int_0^h () \, dz$ $U, C = $ spatial averages

$\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} + \frac{\partial u''c''}{\partial x} = \frac{\partial}{\partial x} \left( E_x \frac{\partial C}{\partial x} \right) + \frac{1}{h} \left( E_z \frac{\partial C}{\partial z} \right)_0$

$u''c'' = \frac{1}{h} \int_0^h u'' c'' \, dz = 0$ no flux

$J_L = \bar{u''} c'' = $ dispersive mass flux $= \left[ \frac{M}{L^2, T} \right]$
\[ J_L = \bar{u''} \bar{c''} = \text{dispersive mass flux} = \frac{M}{L^2, T} \]

Analogy:
\[ J_L = -E_L \frac{\delta C}{\delta x} \quad \text{E}_L = \text{coeff. of longitudinal dispersion} = \frac{L^2}{T} \]

- neglect: long. diffusion, \( E_x << E_L \)

\[ \frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = E_L \frac{\partial^2 C}{\partial x^2} \quad \text{1-D} \]

change of mean conc. advection spreading by
mean velocity deviations + transverse diffusion

Local balance as seen by moving observer (U):
\[ u'' \frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left( E_z \frac{\partial c''}{\partial z} \right) \quad \text{Taylor (1953, 1954)} \]

transport in x direction by differential velocities
\( \Leftrightarrow \) diffusion in vertical direction by turbulence

"Stretching" \( \Leftrightarrow \) "Homogenization"

With given properties: \( u''(z), E_z(z) \) one can compute \( c'' \cdot \frac{\partial C}{\partial x} \),
and evaluate \( \bar{u''} \bar{c''} \)

This leads to:
\[ E_L = -\frac{1}{h} \int_0^h \int_0^2 \int_0^2 u'' dz \, dz \, dz \]
Applications: 2-D Flows

1. Channel flows:
   \[ u'' \sim \log \text{- profile} \]
   \[ E_L = 5.9 u_r h \]
   \[ E_Z \sim \text{parabolic} \]
   Elder (1959)
   \[ E_y \approx E_x = 0.2 u_r h \]
   \[ E_L \gg E_x \]
   Plan view of a drop of dye diffusing in the turbulent flow in an open channel. Distribution of concentration \( C \), normalized to have a maximum of 10. The flow is to the left; \( h = 1.43 \text{ cm}, x/h = 90 \) (Elder, 1959)

2. Turbulent pipe flow:
   \[ E_L = 10.1 u_r r_o \]
   \[ r_o = \text{pipe radius} \]
   Taylor (1954)

3. Laminar pipe flow:
   \[ E_{L,\text{mol}} = \frac{r_o^2 U^2}{48D} \]
   Taylor (1953)
   General rule for applicability: \[ t > t_M = \text{transverse mixing time (vertical)} \]
   \[ = 0.4 \frac{h^2}{E_t} \]
   \( E_t = \text{transverse diffusivity} \)
Natural Rivers

Data: \( \frac{E_L}{u_i h} = 100 \text{ to } 1000 \) 3-D non-uniformities

Lateral velocity deviations dominate

\[ \rightarrow \text{ Stretching} + \]

Lateral (transverse) diffusion

Typical cross section of a natural stream (Fischer 1968)

Fischer (1965) \( \bar{u}(y), E_y(y) \), \( A = \text{cross-section} \)

\[
E_L = \frac{1}{A} \int_0^B (\bar{u} - U) h \int_0^y \frac{1}{E_y h_0} \int_0^y (\bar{u} - U) h \, dy \, dy \, dy
\]

\( u(y): \text{measurements} \) or numerical model

velocity deviation in lateral direction \( y \)

- approximate formula (typical conditions):

\[
E_L = 0.011 \frac{U^2 B^2}{u_i h}
\]
good within factor 4 \( (\sigma_L \sim \text{within } 2) \)

Applicability: \( t > t_m = \text{transverse (lateral mixing time) } = 0.4 \frac{(B/2)^2}{E_y} \)

Other physical non-uniformities:

Overall effects:

- stretching of "cloud"
- increased longitudinal dispersion \( E_L \uparrow \)
Practical longitudinal dispersion equation

\[
\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = E_L \frac{\partial^2 C}{\partial x^2} - kC
\]

\[t > t_M\]

(often neglected; but careful!)

Instantaneous Area Source:

\[
C = \frac{m''}{\sqrt{4\pi E_L t}} \exp \left[ \frac{(x - Ut)^2}{4E_L t} \right] \exp[-kt]
\]

Continuous Area source:

\[
C = \frac{q''}{u \sqrt{1 + \frac{4kE_L}{u^2}}} \exp \left[ \frac{xU}{2E_L \left( \frac{1}{1 + \frac{4kE_L}{u^2}} \right)} \right]
\]

In continuous problems
→ long. Dispersion usually negligible

Plug flow: \[\frac{4kE_L}{U^2} \to 0\]

\[
C = \frac{q''}{U} \exp \left( \frac{-kx}{U} \right)
\]
OCEANIC DIFFUSION  (coastal zone, estuaries, large lakes)

- eddy structure

\[ E_r = C \varepsilon^{1/3} \sigma_r^{4/3} \]

"Richardson’s (1921) 4/3 Law"

\[ \frac{d\sigma_r^2}{dt} = 4E_r \]

\[ \sigma_r^2 \sim \varepsilon t^3 \]  

\[ E_r \sim \varepsilon t^2 \]

\( \neq \) const.!, diffusivity increases with time
Data: \[ \sigma^2_r \sim t^{2.3} \]

\[ E_r \sim \sigma_r^{1.1} \]

Okubo (1971) \( \rightarrow \) Oceanic Diffusion Diagrams

Typical growth rates:

- 1 hr \( \sim \sigma_r = 50 \text{ m} \)
- 1 day \( \sim 1 \text{ km} \)
- 1 week \( \sim 10 \text{ km} \)
- 1 month \( \sim 100 \text{ km} \)

Applications:

Instantaneous Point Source (Accident):

\[ c_{\text{max}} \sim \frac{1}{\sigma^2_r} \quad 2\text{-D growth} \]

(vertically mixed)

\[ c = \frac{M}{2\pi \sigma_r} e^{-\left(\frac{r^2}{2\sigma_r^2}\right)} \]
Continuous release (Routine problems):

\[ U \frac{\partial c}{\partial x} = \frac{\partial}{\partial y} \left( E_y \frac{\partial c}{\partial y} \right) - kc \]

Brooks (1960)

\[ E_y = \frac{E_{y_0}}{b_0} \left( \frac{L}{b_0} \right)^{4/3} \quad 4/3 \text{ Law} \]

\[ L = 2\sqrt{3}\sigma_y \quad \text{22\% width} \]

\[ \frac{L}{b} = \left[ 1 + \frac{8E_{y_0}x^{3/2}}{Ub_0^2} \right] \sim x^{3/2} \]

\[ \frac{c_{\text{max}}}{c_0} = \text{erf} \sqrt{\frac{3/2}{\left( 1 + \frac{8E_{y_0}x^3}{b_0 Ub_0^2} \right) - 1}} \]
References:  Diffusion and Dispersion Processes


Rutherford, J.C., 1994, "River Mixing", John Wiley & Sons, Chichester


